▲ From Number to Variable: Transition to Algebraic Factoring

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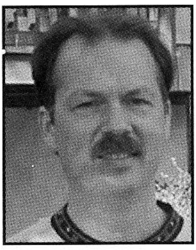

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The middle grades are an important transitional period toward more formal approaches in mathematics in high school. Hirsch and Lappan (1989) identify three important dimensions in which this transition must occur: (1) from number to variable, (2) from specifics to generalizations, and (3) from description to proof. In this article, I will focus on the first dimension of transition, from number to variable, as well as the second dimension, from specifics to generalizations. I will view a transition toward algebraic factoring in two ways: a) as the study of patterns and relationships and b) as generalized arithmetic (see Fernandez and Anhalt, 2001). I propose a method that employs students' number sense and their knowledge of patterns with whole numbers and integers. While this method is intended for middle grades, it can be used with high school students, who do not have a grasp on algebraic factoring, even though they have been taught how to factor before. It is not the intention of this article to provide a shortcut to algebraic factoring. Rather, this method aims at a conceptional level of understanding factoring.

Prerequisite Knowledge

I will begin by providing an inventory of skills and knowledge that should be in place for the method to

work successfully. Students need to be able to factor negative integers and positive integers, using the rules for multiplying integers. For example, 17 factors as 1 x 17 and –1 x –17, while –17 factors as –1 x 17 and 1 x –17. Students must have well-developed ideas about patterns and relationships, a) within a sequence of numbers, and b) between two sequences of numbers. Within a sequence, students should be able to recognize common differences and they should be able to extend a pattern with this knowledge. They should also be able to recognize that all elements in a sequence may have a common factor, or a greatest common factor. Between sequences, patterns focus on the relationship of a sequence to the natural numbers and later the set of integers. This means that they should be able to generate the nth term of a sequence. Students need to be able to substitute a value for a variable in (linear and quadratic) algebraic expressions and calculate the expression value. Last, students need to be able to factor numbers and be able to choose a factor pair such that a pattern will develop. This is explained in the next section.

Factoring Using Patterns

Students begin this method by calculating the values of the expression for a sequence of chosen (integer) values of the variable in a table. It is important that the chosen values of the variable are in a pattern such as consecutive whole numbers or integers. In the beginning the teacher should choose the values such that he/she does not end up with negative integer expression values. Negative integer values should follow after students become sufficiently familiar with the method. We will begin by investigating the expression: 5x + 10. Let x = 1, 2, 3, 4, and 5 (see Table 1). Next explain to the students that they will factor each expression value in two columns, First Factor and Second Factor. In the case of linear expressions the first factor will be a constant. Students should recognize that in Table 1, all expression values have 5 as their greatest common factor. From this they should determine the second factors.

Table 1

x	5x+10	First Factor	Second Factor	pattern
1	15	5	3	5(1+2)
2	20	5	4	5(2+2)
3	25	5	5	5(3+2)
4	30	5	6	5(4+2)
5	35	5	7	5(5+2)
⋮	⋮	⋮	⋮	⋮
n	5n+10	5	n+2	5(n+2)

Now it is time to develop a relationship between the x-values column and the column of second factors. I have found it helpful when students cover up everything between these columns and look for the pattern across. This establishes a one-to-one correspondence of the variable x with the associated collections of first and second factors. Then they can generalize the factorization as 5x + 10 = 5(x + 2), as can be seen in the column labelled "pattern". I have used n as the general value substituted for x in all the examples I present in this article. After sufficient experiences, students can begin to investigate quadratic expressions and apply the same method. We will begin with quadratic binomials and then proceed to trinomials.

Factoring Quadratic Expressions

Table 2 demonstrates the method for the quadratic binomial. Students will not be able to find a greatest common factor in this set of expression values. Their job now is to choose factor pairs for each expression such that the factors in each column form a *pattern.*

Table 2

x	$x^2 + 2x$	First Factor	Second Factor	pattern
1	3	1	3	1(1+2)
2	8	2	4	2(2+2)
3	15	3	5	3(3+2)
4	24	4	6	4(4+2)
5	35	5	7	5(5+2)
⋮	⋮	⋮	⋮	⋮
n	$n^2 + 2n$	n	n+2	n(n+2)

In the table, start with products that factor in one or two ways only, such as primes, and then do the remaining factorizations so these fit a pattern. For example, 3 = 1 x 3; 15 = 3 x 5; and 35 = 5 x 7. Another agreement, which is not always possible to follow, is that the smallest of the factors goes in the First Factor column. Have students articulate the pattern in words and then in symbolic form. For example, "the first factor column is a one-more pattern starting at 1 and the second factor column is a one-more pattern starting at 3."

This method works well for binomials with common factors in each term as we saw in the first example. The next example, $5x^2 + 10x$, is a combination of examples 1 and 2.

Table 3a						
x	$5x^2 + 10x$	First Factor	Second Factor	First Factor	Second Factor	pattern
1	15	5	3	1	3	5(1)(1+2)
2	40	5	8	2	4	5(2)(2+2)
3	75	5	15	3	5	5(3)(3+2)
4	120	5	24	4	6	5(4)(4+2)
5	175	5	35	5	7	5(5)(5+2)
⋮	⋮	⋮	⋮	⋮	⋮	⋮
n	$5n^2 + 10n$	5		n	n+2	5n(n+2)

Most students will connect this quadratic binomial with the previous linear and quadratic examples and realize that all expression values will have a greatest common factor of 5. This will result in the initial First and Second Factor columns. Stimulate students to try to continue factoring the Second Factor column into two new columns (like example 2) until a *pattern develops*. After some experiences some students may start to see that algebraically a common factor can vary in a pattern, such as demonstrated in Table 3b.

Table 3b				
x	$5x^2 + 10x$	First Factor	Second Factor	pattern
1	15	5	3	5(1)(1+2)
2	40	10	4	5(2)(2+2)
3	75	15	5	5(3)(3+2)
4	120	20	6	5(4)(4+2)
5	175	25	7	5(5)(5+2)
⋮	⋮	⋮	⋮	⋮
n	$5n^2 + 10n$	5n	n+2	5n(n+2)

In Table 4, the process is applied to the expression $x^2 - 49$, a difference of squares. This will introduce factoring over the integers. It may not be easy to determine which factor pairs to use, since some numbers in the expression column can be factored in several ways. It may be helpful to point out to students to decide first how 33 and 45 can be factored. Next, choose the best place for the negative sign, again emphasizing the need for a pattern that leads to a generalization. You may need to extend the table beyond x = 5 for some students who have difficulty seeing the relationship between the x-values and the First Factor values. Notice, however, that this problem is an excellent opportunity to involve students' knowledge of adding and subtracting integers, which is supported here by looking for patterns. It is important that when students complete the factor columns via discovery of patterns, they should check that the factors obtained multiply correctly to the corresponding expression value. This may invalidate an inferred pattern or confirm it.

Table 4				
x	$x^2 - 49$	First factor	Second Factor	pattern
1	-48	-6	8	(1–7)(1+7)
2	-45	-5	9	(2–7)(2+7)
3	-40	-4	10	(3–7)(3+7)
4	-33	-3	11	(4–7)(4+7)
5	-24	-2	12	(5–7)(5+7)
⋮	⋮	⋮	⋮	⋮
n	$n^2 - 49$	n – 7	n + 7	(n–7)(n+7)

From Specifics to Generalization

After some practice, it is important to discuss with the students that the algebraic factorization gives information about how a specific set of products generated by an expression can always be decomposed in two or more factors in a patterned manner. The idea of generalization is what is powerful here and often not understood by students. It reveals the structure of algebra very clearly and makes a conceptual link between factoring with numbers and the patterned factoring in algebra.

Factoring Trinomials

The proposed process also works well for trinomials (see Tables 5 and 6), even with the coefficient for x^2 being greater than 1 or negative (see Table 7). The example in Table 5 is interesting, because there appears to be a common factor of two. In fact, all expression values are even.

Table 5				
x	$x^2 + 5x + 6$	First Factor	Second Factor	pattern
1	12	3	4	(1+2)(1+3)
2	20	4	5	(2+2)(2+3)
3	30	5	6	(3+2)(3+3)
4	42	6	7	(4+2)(4+3)
5	56	7	8	(5+2)(5+3)
⋮	⋮	⋮	⋮	⋮
n	$n^2 + 5n + 6$	n+2	n+3	(n+2)(n+3)

This can be proved by looking at the algebraic factors: if x is even, then x + 2 is even and x + 3 is odd, thus their product is even. If x is odd, then x + 2 is odd and x + 3 is even, therefore the product is also even. As you can see in the factor columns, the common factor 2 is not consistently in the same column, and therefore it

should not be factored out here. It will remove the pattern in both columns. To avoid this you may want to start with an expression such as $x^2 + 4x + 3$. This methods also works very well for perfect squares such as $x^2 + 4x + 4$ and $x^2 - 6x + 9$.

The example in Table 6 again allows for students to consider factors of negative products and patterns with integers. You can start a discussion with students about how they could avoid the occurrence of negative products by choosing the values of x carefully, in this case starting at x = 3.

Table 6

x	$x^2 - 2x - 3$	First Factor	Second Factor	pattern
1	-4	2	-2	(1+1)(1-3)
2	-3	3	-1	(2+1)(2-3)
3	0	4	0	(3+1)(3-3)
4	5	5	1	(4+1)(4-3)
5	12	6	2	(5+1)(5-3)
⋮	⋮	⋮	⋮	⋮
n	$n^2 - 2n - 3$	n+1	n-3	(n+1)(n-3)

In Table 7, you can observe that the first two expression values have straightforward factorizations and may establish the pattern. All first factors increase by one, starting at 2, and all second factors are consecutive odd numbers, starting at 5.

Table 7

x	$2x^2 + 5x + 3$	First Factor	Second Factor	pattern
1	10	2	5	(1+1)(2x1+3)
2	21	3	7	(2+1)(2x2+3)
3	36	4	9	(3+1)(2x3+3)
4	55	5	11	(4+1)(2x4+3)
5	78	6	13	(5+1)(2x5+3)
⋮	⋮	⋮	⋮	⋮
n	$2n^2 + 5n + 3$	n+1	2n+3	(n+1)(2n+3)

Therefore, the factorization of the expression is (x + 1)(2x + 3), as can be noted in the pattern column. But this may not be obvious for all students. This may require some of the additional strategies described in the next section. This method works for any quadratic expression, but as you can see from the last example, it requires increasing sophistication with patterns when factors are of forms other than x + a.

Connecting with the Function Concept

Up to this point the method proposed has tried to link with students' number sense and their understanding of patterns within and between sequences of numbers. For more complex situations you may want to use the area model, often demonstrated with Algebra tiles (see, for example, http://www.ucs.mun.ca/~mathed/t/rc/alg/tiles/tiles1.html). However, I propose to look at these situations from a graphical/function point of view. If we view the quadratic expression as a function, then it can be thought of as the product of two linear functions. This approach is readily available to middle grades students by using graphing calculators or spreadsheet programs.

To take the function view, it is necessary for students to have knowledge of linear functions, and the ideas of slope and y-intercept. They should be able to graph linear functions. This can be done on a graphing calculator, entering the table values and using line of best fit, or a spreadsheet, entering the table values and using a trend line capability, to make the connection with graphical representation. This means that students can try to fit a linear equation with each column of factors and the column of chosen x-values. If students include 0 in their choices for x, the y-intercept will roll out for free and after determining the slope the factors are then of the form mx + b. If you do not include 0 for x, as done above, you can use the point-slope formula when two points are known. I will first illustrate this with the example in Table 7 and then I will give an example where we can get to the slope, y-intercept form more directly.

For the first factor, I can choose two points of the form (x, first factor), (1,2) and (2,3). Using the point-slope formula will yield y = x + 1. Therefore, (x + 1) represents the general first factor. Using (1,5) and (2,7) for the second factor will yield y = 2x + 3. Therefore, (2x + 3) represents the general second factor. One of the things that need to be emphasized is that the same general factor must come

Table 8

x	$6x^2 + x - 2$	First Factor	Second Factor	Pattern
-3	49	-7	-7	(2x-3-1)(3x-3+2)
-2	20	-5	-4	(2x-2-1)(3x-2+2)
-1	3	-3	-1	(2x-1-1)(3x-1+2)
0	-2	-1	2	(2x0-1)(3x0+2)
1	5	1	5	(2x1-1)(3x1+2)
2	24	3	8	(2x2-1)(3x2+2)
⋮	⋮	⋮	⋮	⋮
n	$6n^2 + n - 2$	2n-1	3n+2	(2n-1)(3n+2)

from any choice of two points. If students get several different general factors for arbitrary choices of pairs of points, they must conclude that there is no general factorization possible for that quadratic expression.

Table 8 is an example where I chose 0 as one of the x-values and where patterns are maybe more difficult to discern.

For the first factor you can see that the y-intercept is -1 and the slope is 2, making $(2x - 1)$ the first factor, and that similarly the second factor is $(3x + 2)$. This appears to be easier. However you may not be able to avoid having to deal with negative factors. I found that spreadsheets and graphing calculators that allowed for curve fitting generated these expressions very easily as well. As stated earlier, the tabular format forms a good transition to using this technology.

Showing that Factoring Cannot be Done

Students can also decide that an expression *can not* be factored by means of this technique. Let the expression be $x^2 + x + 1$. Then the table looks as follows:

Table 9

x	$x^2 + x + 1$	first factor	second factor
1	3	1	3
2	7	1	7
3	13	1	13
4	21	1 or 3	21 or 7
5	31	1	31

Only one of the values in the second column can be factored other than using primes. Therefore, there is no general factorization possible of this trinomial. This method does not distinguish between non-factorable expressions whose correspondng equation has irrational roots, for example $x^2 + x = -1$, and those that have complex roots, such as the above example.

Conclusions

You can continue to experiment further with this method before you take it to your class. For example, this method extends to simplifying rational expressions. You are encouraged to try this method for yourself and determine whether it is a useful strategy in your classroom. It worked well for me when I did this with remedial high school students, who had not learned how to factor after two years of formal instruction. Experiment with choosing x-values such that negative product values show up. Does this method work with expressions of degree other than 2? How could you employ a spreadsheet or a graphing calculator with this method?

I have found that students' development in learning algebra hits a symbolic wall. When I returned to my students' number sense and their sense of pattern, they could discover how each algebraic expression generates a special set of numbers that can be factored in a patterned way. This provided meaning to their learning of algebra. It was not until after the algebra was meaningful to my students that they were open to using other ways of investigating these problems. I helped my students derive meaning by connecting with what they knew well.

References

Fernandez, M, and Anhalt, C, (2001). Transition toward algebra. *Mathematics Teaching in the Middle, 7*(4), 236-241.

Hirsh, R, and Lappan, G, (1989). Transition to high school mathematics. *Mathematics Teacher, 82*, 614-618. ▲

▲ Upcoming Events

For further information, please contact the Chapter Representative (see last page of the *Gazette*) for OAME conferences, and Frances Schatz, OAME's NCTM Representative, for NCTM conferences.

OAME Conferences
(all Chapter Conferences are held after school)

October 16, 2003	Markham	Y⁴MA Chapter Conference
October 17 – 18, 2003	Sudbury	NOMA Chapter Conference
October 23, 2003	Orillia	MAC² Chapter Conference
November 5, 2003	*location yet to be determined*	ISOMA Chapter Conference
November 8, 2003	Ottawa	COMA Chapter Conference
November 13, 2003	Kitchener	GVMA Chapter Conference
Early November *(exact date yet to be determined)*	Kingston	QSLMA Chapter Conference
Early February *(exact date yet to be determined)*	Thunder Bay	NWOAME Chapter Conference
April 28 – May 1, 2004	University of Waterloo	31st Annual OAME Conference

NCTM Conferences

November 20 – 22, 2003	Edmonton, Alberta	Canadian Regional Conference
April 21 – 24, 2004	Philadelphia, Pennsylvania	82nd Annual Meeting

Other

October 3 – 4, 2003	Brampton	Texas Instruments Regional Conference

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